BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 10

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CONTENTS

I. REVIEW

Last time we:

(1) Reviewed definitions and results on covering spaces and the universal covering space of a Riemann surface. Described the Galois correspondence between covers of a Riemann surface X and subgroups of $\pi_1(X, x)$.
(2) Defined properties of the action of a group G on Riemann surface X and studied the resulting properties of the quotient $G \backslash X$ and quotient map $\pi : X \to G \backslash X$.

II. MORE MONODROMY

Recall that a transitive subgroup of S_d is one that acts transitively on $\{1, 2, \ldots, d\}$.

II.1. Morphisms to monodromy representations.

Lemma 1. *Let X, Y be topological spaces, let $p : X \to Y$ be a covering map of finite degree, and let $\rho : \pi_1(Y, y) \to S_d$ be its associated monodromy representation. If X is path-connected, then the image of ρ is a transitive subgroup of S_d.*

Proof. Fix indices i and j, and let x_i, x_j be the corresponding points in the fiber $p^{-1}(y)$. Since X is path-connected, then there exists a path δ starting x_i and ending at x_j. Letting $\gamma = p \circ \delta$, then γ is a loop in Y based at y. Moreover, by uniqueness the lift of γ starting at x_i must be δ, so $\rho([\gamma])$ maps i to j. □

Example 2. Let $\mathfrak{D}^* := \mathfrak{D} \setminus 0$ be the punctured open unit disc, considered as a subset of $\mathbb{C}$. Let $p : \mathfrak{D}^* \to \mathfrak{D}^*$ be the covering map given by $w \mapsto w^d$ for some $d \in \mathbb{Z}_{\geq 1}$. Take $z_0 = 1/2^d$ as the basepoint of the codomain. Letting ζ be an primitive d^{th} root of unity, then $p^{-1}(z_0)$ consists of the points $w_j := \zeta^j/2$ for $j = 1, \ldots, d$.

Date: March 31, 2021.

Letting $\gamma : [0,1] \to \mathfrak{D}^*$, $\gamma(t) = \dfrac{1}{2^d} e^{2\pi i t}$, then $[\gamma]$ is a generator for $\pi_1(\mathfrak{D}^*, z_0)$. The loop γ lifts to the loops $\widetilde{\gamma}_j : [0,1] \to \mathfrak{D}^*$ given by $\widetilde{\gamma}_j(t) = \zeta^j \dfrac{1}{2} e^{2\pi i t/d}$, whose starting point is $w_j = \zeta^j/2$ and whose ending point is $w_{j+1} = \zeta^{j+1}/2$. Thus the monodromy representation $\rho : \pi_1(\mathfrak{D}^*, z_0) \to S_d$ sends $[\gamma]$ to the cyclic permutation that takes j to $j+1$, i.e.,

$$\rho([\gamma]) = (1\ 2\ \cdots\ d)\,.$$

We now discuss the monodromy of a morphism $F : X \to Y$ of degree d of compact, connected Riemann surfaces. By the Local Normal Form theorem, every morphism of Riemann surfaces locally looks like $z \mapsto z^d$, so our above example is actually quite general. Let $\Sigma \subseteq Y$ be its set of ramification values and let $Y^* := Y \setminus \Sigma$ and $X^* := X \setminus F^{-1}(\Sigma)$. As we saw previously, then the restriction $F|_{X^*} : X^* \to Y^*$ is an (unramified) covering map. The monodromy representation of F is defined to be the monodromy representation $\rho : \pi_1(Y^*, y) \to S_d$ of this resriction. Since X is connected, then $\mathrm{img}(\rho) \leq S_d$ is a transitive subgroup.

Lemma 3. *With notation as above, suppose above a ramification value $b \in Y$ there are k preimages $u_1, \ldots, u_k \in F^{-1}(b)$, with ramification indices $e_i := e_{u_i}(F)$. Then the permutation σ representing a small loop around b has cycle structure $(e_1, \ldots, e_k)$, i.e., it is composed of k disjoint cycles of lengths $e_1, \ldots, e_k$.*

Proof. Let $y \in Y$ be a basepoint. Fix a ramification value $b \in Y$ and choose a small open neighborhood W of b that is isomorphic to the open disc $\mathfrak{D}$. Let $u_1, \ldots, u_k$ be the points in the fiber $F^{-1}(b)$; since b is a ramification value, then at least one of the u_j must be a ramification point.

Choose W small enough such that $F^{-1}(W \setminus \{b\})$ decomposes as a disjoint union of open punctured neighborhoods $U_1^*, \ldots, U_k^*$ of $u_1, \ldots, u_k$, respectively. Let $U_j = U_j^* \cup \{u_j\}$ be the corresponding (non-punctured) open neighborhood of u_j. Letting $e_j = e_{u_j}(F)$, then by the Local Normal Form Theorem, there are coordinates z_j on U_j and z on W such that F locally has the form $z_j \mapsto z_j^{e_j}$.

Then F sends $U_j \setminus \{u_j\}$ to $W \setminus \{b\}$ via the e_j^{th} power map. Choose a path α from the basepoint y to a point $y_0 \in W \setminus \{b\}$, and let β be a loop in $W \setminus \{b\}$ based at y_0 that winds once around the ramification value b. Then the path $\gamma := \alpha^{-1} * \beta * \alpha$ is a loop in Y based at y, which we will call a small loop on Y around b. Since F is an unramified covering away from Σ, then the path α simply gives a bijection between the fibers $F^{-1}(y)$ and $F^{-1}(y_0)$. Thus the permutation σ of the fiber $F^{-1}(y)$ is determined up to this identification by the loop β around b.

Above the punctured neighborhood $W \setminus \{b\}$ we have k punctured discs $U_j \setminus \{u_j\}$, each mapping to $W \setminus \{b\}$ via the e_j^{th} power map. By the example above, the monodromy for each covering $F|_{U_j \setminus \{u_j\}} : U_j \setminus \{u_j\} \to W \setminus \{b\}$ is a cyclic permutation of the e_j preimages of y_0 which lie in U_j. Thus the loop β based at y_0 and hence the loop γ based at y induce cyclic permutations of the points above y, and the cycle corresponding to u_j has length e_j. □

II.2. **Monodromy representations to morphisms.** Let Y be a compact, connected Riemann surface and fix a base point $y \in Y$. Suppose $\rho : \pi_1(Y, y) \to S_d$ is a homomorphism with transitive image. Thus $\pi_1(Y, y)$ acts transitively on $\{1, \ldots, d\}$ via ρ. Fix an index in $\{1, \ldots, d\}$, say 1, and let $H \leq \pi_1(Y, q)$ be the stabilizer of 1:

$$H = \text{Stab}_{\pi_1(Y,y)}(1) = \{[\gamma] \in \pi_1(Y, y) \mid \rho([\gamma])(1) = 1\}$$

Since the image of ρ is transitive, then $[\pi_1(Y, y) : H] = d$ by the Orbit-Stabilizer Theorem. By the Galois correspondence between coverings and subgroups of the fundamental group, then there is a covering space X and a covering map $F : X \to Y$ whose monodromy representation is ρ. Thus we have a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{connected coverings} \\ F : X \to Y \text{ of degree } d \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{group homomorphism} \\ \rho : \pi_1(Y, y) \to S_d \text{ with} \\ \text{transitive image, up to} \\ \text{conjugacy in } S_d \end{array} \right\}.$$

As we saw previously, since Y is a Riemann surface, the covering map $F : X \to Y$ induces a unique holomorphic structure on X such that F is a morphism of Riemann surfaces. Thus on the lefthand side of the above bijection, we can further insist that X is a Riemann surface and F is holomorphic.

Proposition 4. *Let Y be a compact, connected Riemann surface, let B be a finite subset of Y, and let $y \in Y \setminus B$ be a basepoint. Then there is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{morphisms } F : X \to Y \textit{ of} \\ \textit{degree } d \textit{ whose} \\ \textit{ramification values lie in} \\ B \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{group homomorphism} \\ \rho : \pi_1(Y \setminus B, y) \to S_d \\ \textit{withe transitive image, up} \\ \textit{to conjugacy in } S_d \end{array} \right\}.$$

Proof idea. We have already seen that a morphism of Riemann surfaces induces a monodromy representation. To see the reverse association, let ρ be a group homomorphism as above, and let $Y^* = Y \setminus B$. By the bijection above, then there is a covering space X^* and a covering map $\widehat{F} : X^* \to Y^*$ with monodromy representation ρ. One then must argue that we can "fill in the holes" in X^* to obtain a compact, connected Riemann surface X and extend $\widehat{F}$ to a morphism $F : X \to Y$ of Riemann surfaces. □

In the case where $Y = \mathbb{P}^1$, we can be even more explicit, since we know what the fundamental group of $V := \mathbb{P}^1 \setminus \{b_1, \ldots, b_n\}$ is. [What is it for $n = 1, 2, 3$?]

Let γ_j be a small loop around b_j based at y. Then the fundamental group of V is

$$\pi_1(V, y) = \langle [\gamma_1], [\gamma_2], \ldots, [\gamma_n] \mid [\gamma_1][\gamma_2] \cdots [\gamma_n] = 1 \rangle \cong \langle [\gamma_1], [\gamma_2], \ldots, [\gamma_{n-1}] \rangle,$$

the free group on $n - 1$ generators.

Corollary 5. *Let $B = \{b_1, \dots, b_n\} \subseteq \mathbb{P}^1$ be a finite set. Then there is a bijective correspondence*

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{morphisms } F : X \to \mathbb{P}^1 \text{ of} \\ \text{degree } d \text{ whose} \\ \text{ramification values lie in} \\ B \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} n\text{-tuples } (\sigma_1, \dots, \sigma_n) \text{ of} \\ \text{permutations in } S_d \text{ such} \\ \text{that } \sigma_1 \cdots \sigma_n = 1 \text{ and} \\ \langle \sigma_1, \dots, \sigma_n \rangle \leq S_d \text{ is} \\ \text{transitive, up to} \\ \text{simultaneous conjugacy} \end{array} \right\}.$$

Moreover, if σ_j has cycle structure $(e_1, \dots, e_k)$, then b_j has k preimages $u_1, \dots, u_k$ with ramification indices $e_{u_j} = e_j$ for each j.

III. Function fields

I'm not a fan of the presentation of function fields in Girondo and González-Diez, so today we'll be following chapter 8 of Forster's *Lectures On Riemann Surfaces*.

Recall that for a Riemann surface X, $\mathcal{M}(X)$ is the field of meromorphic functions on X. An element $f \in \mathcal{M}(X)$ is a mermorphic function $f : X \to \mathbb{C}$, and we saw that such a function can be viewed as a morphism $X \to \widehat{\mathbb{C}}$.

Given a nonconstant morphism $\pi : Y \to X$ of Riemann surfaces and $f \in \mathcal{M}(X)$, then $f \circ \pi : Y \xrightarrow{\pi} X \xrightarrow{f} \widehat{\mathbb{C}}$ is a morphism, hence can be viewed as a meromorphic function on Y. Thus we get a field morphism

$$\begin{aligned} \pi^* : \mathcal{M}(X) &\to \mathcal{M}(Y) \\ f &\mapsto f \circ \pi \,. \end{aligned}$$

We often consider $\mathcal{M}(X)$ as a subfield of $\mathcal{M}(Y)$ by identifying it with its image $\pi^*(\mathcal{M}(X))$.

Example 6. Let $Y = X = \widehat{\mathbb{C}}$ and consider the morphism

$$\begin{aligned} \pi : \widehat{\mathbb{C}} &\to \widehat{\mathbb{C}} \\ z &\mapsto z^3 \,. \end{aligned}$$

Given a meromorphic function $f \in \mathcal{M}(X)$, then $\pi^*(f)(z) = f \circ \pi(z) = f(z^3)$. Thus the corresponding extension of function fields is $\mathbb{C}(z) \supseteq \mathbb{C}(z^3)$.

$$\begin{array}{ccc} Y = \widehat{\mathbb{C}} & \qquad & \mathbb{C}(z) \\ {\scriptstyle \pi}\downarrow & & \Big| {\scriptstyle \pi^*} \\ X = \widehat{\mathbb{C}} & & \mathbb{C}(z^3) \end{array}$$

Remark 7. The associations $X \mapsto \mathcal{M}(X)$, $\pi \mapsto \pi^*$ define a contravariant functor from the category of Riemann surfaces to the category of fields. We will later see that this is in fact an equivalence when we restrict the target category to function fields of one variable.

Let X and Y be Riemann surfaces and $\pi : Y \to X$ be an unramified covering map of degree d, and let $f \in \mathcal{M}(Y)$ be a meromorphic function. Then each point $x \in X$

has an evenly covered open neighborhood U, so $\pi^{-1}(U) = \bigsqcup_{j=1}^{d} V_d$, and the restrictions $\pi|_{V_j} : V_j \to U$ are isomorphisms of Riemann surfaces. Let $\tau_j : U \to V_j$ be the inverse of $\pi|_{V_j}$, and let

$$f_j := \tau_j^* f = f \circ \tau_j \in \mathcal{M}(U)\,.$$

Let T be a variable and define a polynomial in $\mathcal{M}(U)[T]$ by

$$\prod_{j=1}^{d}(T - f_j) = T^n + c_1 T^{n-1} + \cdots + c_d\,.$$

Then the c_j are meromorphic functions on U and

$$c_j = (-1)^j s_j(f_1, \ldots, f_n)$$

where s_j is the j^{th} elementary symmetric functions in d variables. One can show that these c_j are independent of the choice of neighborhood U of x. Thus we can cover X with evenly covered neighborhoods and glue together the locally defined c_j to obtain meromorphic functions $c_1, \ldots, c_d \in \mathcal{M}(X)$ on all of X.

One can show that these signed symmetric functions can also be defined when $\pi : Y \to X$ is a nonconstant morphism of Riemann surfaces (which may have ramification points). The strategy to prove this is one we've seen before: throw out the ramification values and their preimages to obtain an unramified covering map $\pi|_{Y^*} \to X^*$. Since this restriction is unramified, then we know that the signed symmetric functions c_j of f exist in this case. Then one argues that these c_j can be meromorphically continued to the ramification values.

Theorem 8. *Suppose X and Y are Riemann surfaces and $\pi : Y \to X$ is a morphism of degree d. If $f \in \mathcal{M}(Y)$ and $c_1, \ldots, c_d \in \mathcal{M}(X)$ are the signed elementary symmetric functions of f, then*

$$f^d + (\pi^* c_1) f^{d-1} + \cdots + (\pi^* c_{d-1}) f + \pi^* c_d = 0\,. \tag{1}$$

The monomorphism $\pi^* : \mathcal{M}(X) \to \mathcal{M}(Y)$ is an algebraic field exntesion of degree d.

Proof. We'll just show that $[\mathcal{M}(Y) : \mathcal{M}(X)] \leq d$; to show equality requires the fact that the functions on a Riemann surface separate points. We will show that the lefthand side of (1) is the zero function on Y. Given $y \in Y$, then there exists some $k \in \{1, \ldots, d\}$ such that $\tau_k \circ \pi(y) = y$. Then

$$\begin{aligned}(f^d + (\pi^* c_1) f^{d-1} + \cdots + (\pi^* c_{d-1}) f + \pi^* c_d)(y) &= \prod_{j=1}^{d}(f(y) - \pi^* f_j(y)) \\ &= \prod_{j=1}^{d}(f(y) - f \circ \tau_j \circ \pi(y)) = 0\end{aligned}$$

since one of the factors is

$$f(y) - f \circ \tau_k \circ \pi(y) = f(y) - f(y) = 0\,.$$

Thus every element $f \in \mathcal{M}(Y)$ is the root of a polynomial in $\mathcal{M}(X)[T]$ of degree at most d, so $[\mathcal{M}(Y) : \mathcal{M}(X)] \leq d$. ☐

Definition 9. A covering map $p : Y \to X$ of topological spaces is Galois or normal if for every points $y_0, y_1 \in Y$ with $p(y_0) = p(y_1)$ (i.e., in the same fiber) there exists a deck transformation $\sigma : Y \to Y$ such that $\sigma(y_0) = y_1$.

Remark 10.

- In other words, the covering is Galois if the group of deck transformations acts transitively on every fiber. This is analogous to the case of fields. Let $f \in F[x]$ be an irreducible polynomial, and let $K = F(\alpha)$ where α is a root of f. The extension K/F of fields is Galois iff $\mathrm{Aut}(K/F)$ acts transitively on the roots of f.
- There is an alternative characterization of Galois covering maps using the fundamental group. Choose a basepoint $y_0 \in p^{-1}(x_0)$. Then the covering map $p : Y \to X$ induces a group homomorphism $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ The cover $p : Y \to X$ is Galois iff $p_*\pi_1(Y, y_0)$ is a normal subgroup of $\pi_1(X, x_0)$.

Example 11. Let $Y = X = \mathbb{C}^\times$, and consider the covering map

$$\begin{aligned} p : Y &\to X \\ z &\mapsto z^3 . \end{aligned}$$

Then $\mathrm{Deck}(Y/X) = \{\mathrm{id}_Y, \sigma, \sigma^2\}$, where

$$\begin{aligned} \sigma : Y &\to Y \\ z &\mapsto \zeta z , \end{aligned}$$

where ζ is a primitive third root of unity. Given $x_0 \in X$, then

$$p^{-1}(x_0) = \{y_0, y_1, y_2\} = \{\sqrt[3]{x_0}, \zeta\sqrt[3]{x_0}, \zeta^2\sqrt[3]{x_0}\}$$

where $y_j = \zeta^j\sqrt[3]{x_0}$. Thus we see that $\mathrm{Deck}(Y/X)$ acts transitively on $p^{-1}(x_0)$: for instance, $\sigma(y_0) = y_1$, and $\sigma^2(y_0) = y_2$. Thus $p : Y \to X$ is Galois.

We can extend the notion of Galois to nonconstant morphisms of Riemann surfaces. Let $F : Y \to X$ be a nonconstant morphism of Riemann surfaces, and let $R \subseteq X$ be its ramification values. Let $X^* = X \setminus R$ and $Y^* = Y \setminus F^{-1}(R)$. As we have seen, then $F|_{Y^*} : Y^* \to X^*$ is a covering map.

Definition 12. A nonconstant morphism $F : Y \to X$ of Riemann surfaces is Galois or normal if the restricted covering map $F|_{Y^*} : Y^* \to X^*$ is Galois.

Theorem 13. *Let X be a Riemann surface and suppose that*

$$P(T) = T^n + c_1T^{n-1} + \cdots + c_n \in \mathcal{M}(X)[T]$$

is an irreducible polynomial of degree n. Then there exist a Riemann surface Y, a morphsim $F : Y \to X$ of degree n and a mermorphic function $f \in \mathcal{M}(Y)$ such that $(F^*P)(f) = 0$.

Definition 14. Such a triple (Y, F, f) is called the algebraic function defined by $P(T)$.

The proof of this result is the machinery of multi-valued functions. Here's a proof by example for the case $X = \mathbb{A}^1$ with coordinate S. Then the coefficients $c_1, \ldots, c_n$ are simply rational functions in S. Multiplying $P(T)$ by the least common denominator of $c_1, \ldots, c_n$, we obtain a polynomial $Q(S, T) \in \mathbb{C}[S, T]$. Let C be the curve in $\mathbb{A}^2$ given by $Q(S, T) = 0$.

One can show that Q is irreducible, however C may have singular points. One can resolve these singular points and obtain a smooth curve C', which is the desired Riemann surface.

Theorem 15. *Let X be a Riemann surface and $K := \mathcal{M}(X)$ be its field of meromorphic functions. Suppose $P(T) \in K[T]$ is an monic, irreducible polynomial of degree d.*

(a) Let (Y, π, F) be the algebraic function defined by $P(T)$ and let $L := \mathcal{M}(Y)$. Identifying K with its image under $\pi^* : K \to L$, then L/K is field extension of degree d and $L \cong K[T]/(P(T))$.

(b) Every deck transformation $\sigma \in \operatorname{Deck}(Y/X)$ induces an automorphism

$$\begin{aligned} \widehat{\sigma} : L &\to L \\ f &\mapsto \sigma \cdot f := f \circ \sigma^{-1} \end{aligned}$$

that fixes K, and the map

$$\begin{aligned} \operatorname{Deck}(Y/X) &\to \operatorname{Aut}(L/K) \\ \sigma &\mapsto \widehat{\sigma} \end{aligned}$$

is a group homomorphism, and in fact, an isomorphism.

(c) The covering $Y \to X$ is Galois if and only if the extension L/K of function fields is Galois.

Proof. (a) The first statement follows from results above.

(b) To show that the map $\operatorname{Deck}(Y/X) \to \operatorname{Aut}(L/K)$ is an isomorphism requires more results about deck transformations, but we can at least show it's a homomorphism. Given $\sigma, \tau \in \operatorname{Deck}(Y/X)$ and $f \in \mathcal{M}(Y)$, then

$$\begin{aligned} \widehat{\sigma \circ \tau}(f) &= (\sigma \circ \tau) \cdot f = f \circ (\sigma \circ \tau)^{-1} = f \circ \tau^{-1} \circ \sigma^{-1} = \widehat{\sigma}(f \circ \tau^{-1}) \\ &= \widehat{\sigma}(\widehat{\tau}(f)) = \widehat{\sigma} \circ \widehat{\tau}(f) \end{aligned}$$

so $\widehat{\sigma \circ \tau} = \widehat{\sigma} \circ \widehat{\tau}$.

(c) (Assume we know part (b) is true.) Fix a basepoint $x_0 \in X$, and let $\pi^{-1}(x_0) = \{y_1, \ldots, y_d\}$. Then $\pi : Y \to X$ is Galois iff for each $j = 1, \ldots, d$ there exists a deck transoformation $\sigma_j \in \operatorname{Deck}(Y/X)$ such that $\sigma_j(y_1) = y_j$. (This is glossing over some details, but they follow from uniqueness of lifts for covering maps.) Thus $\pi : Y \to X$ is Galois iff $\# \operatorname{Deck}(Y/X) = d$. Similarly, L/K is Galois iff $\# \operatorname{Aut}(L/K) = [L : K] = d$. Since $\operatorname{Deck}(Y/X) \cong \operatorname{Aut}(L/K)$ by the previous part, then the result follows.

□

Example 16. Let $E : y^2 = x^3 - x$ be an elliptic curve and $\pi : E \to \mathbb{P}^1$ be the projection $(x, y) \mapsto x$. Then the corresponding extension of function fields is

$$\begin{array}{ccc} E & \qquad & \mathcal{M}(E) = \dfrac{\mathbb{C}(x)[y]}{(y^2 - (x^3 - x))} \\ \Big\downarrow \pi & & \Big| \, \pi^* \\ \mathbb{P}^1 & & \mathbb{C}(x)\,. \end{array}$$

The deck transformations of π are the identity and the hyperelliptic involution $\iota : (x, y) \mapsto (x, -y)$, and the corresponding field automorphism is

$$\begin{aligned} \iota^* : \mathcal{M}(E) &\to \mathcal{M}(E) \\ f(x,y) &\mapsto f(x,-y)\,. \end{aligned}$$

Thus we see that the covering is Galois: given $x_0 \in \mathbb{P}^1$, the points in the fiber π^{-1} are simply (x_0, y_0) and $(x_0, -y_0)$, where y_0 is a solution to $y^2 - (x_0^3 - x_0)$, and these points are exchange by the involution ι.

We also see directly that the extension of function fields is Galois, as $[\mathcal{M}(E) : \mathbb{C}(x)] = 2$ and $\mathrm{Aut}(\mathcal{M}(E)/\mathbb{C}(x)) = \{\mathrm{id}, \iota^*\}$, so $\# \mathrm{Aut}(\mathcal{M}(E)/\mathbb{C}(x)) = 2$.

Definition 17. A function field in one variable is a finite extension of $\mathbb{C}(x)$, the field of rational functions with coefficients in $\mathbb{C}$.

Proposition 18. *There is an equivalence of categories between the category of compact, connected Riemann surfaces, whose arrows are morphisms of Riemann surfaces, and the category of function fields in one variables, whose arrows are field monomorphisms.*